COMPLEXITY CLASSES EXAMPLES

(download slides and .py files to follow along)

6.100L Lecture 23

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THETA

- **Theta Θ** is how we denote the **asymptotic complexity**
- We look at the **input term that dominates** the function
  - Drop other pieces that don't have the fastest growth
  - Drop additive constants
  - Drop multiplicative constants
- End up with only a **few classes of algorithms**
- We will look at code that lands in each of these classes today

WHERE DOES THE FUNCTION COME FROM?

- Given code, start with the input parameters. What are they?
- Come up with the equation relating input to number of ops.
  - `f = 1 + len(L1)*5 + 1 + len(L2)*5 + 2 = 5*len(L1) + 5*len(L2) + 3`
  - If lengths are the same, `f = 10*len(L) + 3`
- Θ(f) = Θ (10*len(L) + 3) = Θ(len(L))

Only care about code that repeats wrt these variables

```
def f(L, L1, L2):
    inL1 = False
    for i in range(len(L1)):
        if L[i] == L1[i]:
            inL1 = True
    inL2 = False
    for i in range(len(L2)):
        if L[i] == L2[i]:
            inL2 = True
    return inL1 and inL2
```

Loop repeats as a function of input

Loop repeats as a function of input

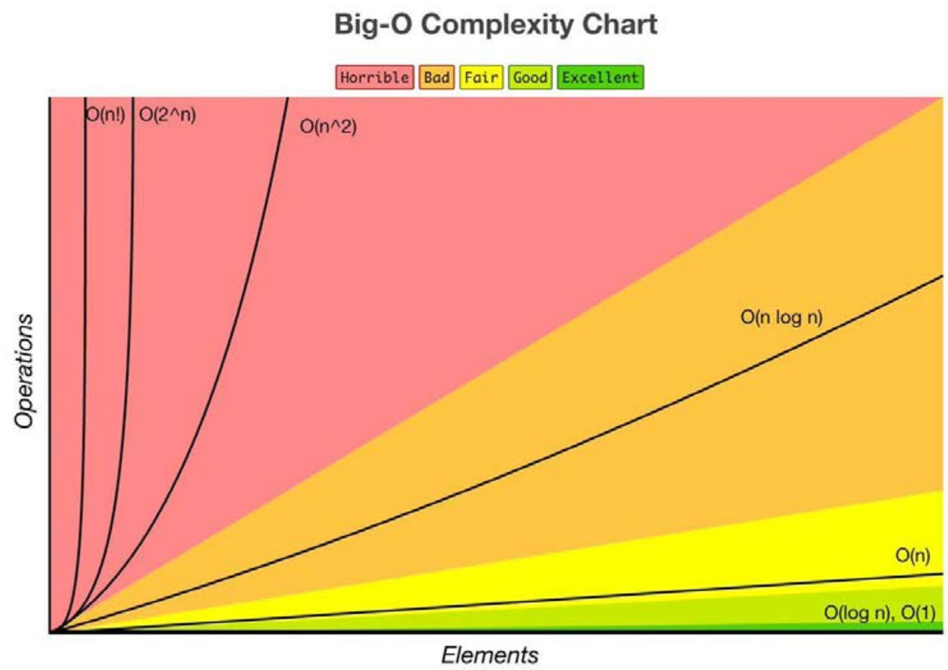
WHERE DOES THE FUNCTION COME FROM?

- A quicker way: no need to come up with the exact formula. Look for loops and anything that repeats wrt the input parameters. Everything else is constant.

Only care about code that repeats wrt these variables

```
def f(L, L1, L2):
    inL1 = False
    for i in range(len(L1)):
        if L[i] == L1[i]:
            inL1 = True
    inL2 = False
    for i in range(len(L2)):
        if L[i] == L2[i]:
            inL2 = True
   return inL1 and inL2
```

COMPLEXITY CLASSES
n is the input



We want to design algorithms that are as close to top of this hierarchy as possible

- *Θ(1)* denotes **constant** running time
- *Θ(log n)* denotes **logarithmic** running time
- *Θ(n)* denotes **linear** running time
- *Θ(n log n)* denotes **log-linear** running time
- $\Theta(n^c)$ denotes **polynomial** running time
  (c is a constant)
- $\Theta(c^n)$ denotes **exponential** running time
  (c is a constant raised to a power based on input size)

CONSTANT COMPLEXITY

CONSTANT COMPLEXITY

- Complexity **independent of inputs**
- Very few interesting algorithms in this class, but can often have pieces that fit this class
- **Can have loops or recursive calls**, but number of iterations or calls independent of size of input
- Some built-in operations to a language are constant
  - Python indexing into a list `L[i]`
  - Python list append `L.append()`
  - Python dictionary lookup `d[key]`

CONSTANT COMPLEXITY: EXAMPLE 1

```
def add(x, y):
    return x+y
```

- Complexity in terms of either x or y: **$\Theta(1)$**

CONSTANT COMPLEXITY: EXAMPLE 2

```
def convert_to_km(m):
    return m*1.609
```

- Complexity in terms of m: **Θ(1)**

CONSTANT COMPLEXITY: EXAMPLE 3

```
def loop(x):
    y = 100
    total = 0
    for i in range(y):
        total += x
    return total
```

- Complexity in terms of x (the input parameter): **Θ(1)**

LINEAR COMPLEXITY

LINEAR COMPLEXITY

- Simple **iterative loop** algorithms
  - Loops must be a **function of input**
- Linear search a list to see if an element is present
- Recursive functions with **one recursive call and constant overhead** for call
- Some built-in operations are linear
  - `e in L`
  - Subset of list: e.g. `L[:len(L)//2]`
  - `L1 == L2`
  - `del(L[5])`

COMPLEXITY EXAMPLE 0 (with a twist)

- Multiply x by y

```
def mul(x, y):
    tot = 0
    for i in range(y):
        tot += x
    return tot
```

Choice of input on which to measure complexity matters

- Complexity in terms of y: **Θ(y)**
- Complexity in terms of x: **Θ(1)**

BIG IDEA

Be careful about what the inputs are.

LINEAR COMPLEXITY: EXAMPLE 1

- Add characters of a string, assumed to be composed of decimal digits

```
def add_digits(s):
    val = 0
    for c in s:
        val += int(c)
    return val
```

Loop goes through len(s) times: **Θ(len(s))**

Everything else is constant. **Θ(1)**

- **Θ(len(s))**
- **Θ(n) where n is len(s)**

LINEAR COMPLEXITY: EXAMPLE 2

- Loop to find the factorial of a number >=2

```
def fact_iter(n):
    prod = 1
    for i in range(2, n+1):
        prod *= i
    return prod
```

Loop goes through n-1 times:
Θ(n)

Everything else is constant.
Θ(1)

- Number of times around loop is n-1
- Number of operations inside loop is a constant
  - Independent of n
- Overall just **Θ(n)**

FUNNY THING ABOUT FACTORIAL AND PYTHON

```
iter fact(40) took 3.10e-06 sec (322,580.65/sec)
iter fact(80) took 6.00e-06 sec (166,666.67/sec)
iter fact(160) took 1.34e-05 sec (74,626.87/sec)
iter fact(320) took 3.39e-05 sec (29,498.53/sec)
iter fact(640) took 1.18e-04 sec (8,488.96/sec)
iter fact(1280) took 4.31e-04 sec (2,322.88/sec)
iter fact(2560) took 1.33e-03 sec (752.73/sec)
iter fact(5120) took 4.94e-03 sec (202.24/sec)
iter fact(10240) took 1.90e-02 sec (52.50/sec)
iter fact(20480) took 7.66e-02 sec (13.06/sec)
iter fact(40960) took 3.35e-01 sec (2.99/sec)
iter fact(81920) took 1.60e+00 sec (0.62/sec)
```

- Eventually grows faster than linear
- Because Python increases the size of integers, which yields more costly operations
- For this class: ignore such effects

LINEAR COMPLEXITY: EXAMPLE 3

```
def fact_recur(n):
    """ assume n >= 0 """
    if n <= 1:
        return 1
    else:
        return n*fact_recur(n - 1)
```

Think about the function call stack: **Θ(n)**

Everything else is constant. **Θ(1)**

- Computes factorial recursively
- If you time it, notice that it runs a bit slower than iterative version due to function calls
- **Θ(n)** because the number of function calls is linear in n
- **Iterative and recursive factorial** implementations are the **same order of growth**

LINEAR COMPLEXITY: EXAMPLE 4

```
def compound(invest, interest, n_months):
    total=0
    for i in range(n_months):
        total = total * interest + invest
    return total
```

Θ(n_months)

Θ(1)

- Θ(1)*Θ(n_months) = **Θ(n_months)**
  Θ(n) where n=n_months
  - If I was being thorough, then need to account for assignment and return statements:
  - Θ(1) + 4*Θ(n) + Θ(1) = Θ(1 + 4*n + 1) = **Θ(n) where n=n_months**

COMPLEXITY OF ITERATIVE FIBONACCI

```
def fib_iter(n):
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        fib_i = 0
        fib_ii = 1
        for i in range(n-1):
            tmp = fib_i
            fib_i = fib_ii
            fib_ii = tmp + fib_ii
        return fib_ii
```

constant $\Theta(1)$

constant $\Theta(1)$

linear $\Theta(n)$

constant $\Theta(1)$

$\Theta(1)+ \Theta(1)+ \Theta(n)*\Theta(1)+ \Theta(1)$

$\rightarrow$ **$\Theta(n)$**

POLYNOMIAL COMPLEXITY

POLYNOMIAL COMPLEXITY (OFTEN QUADRATIC)

- Most **common polynomial algorithms are quadratic**, i.e., complexity grows with square of size of input
- Commonly occurs when we have **nested loops** or recursive function calls

QUADRATIC COMPLEXITY: EXAMPLE 1

```
def g(n):
    """ assume n >= 0 """
    x = 0
    for i in range(n):
        for j in range(n):
            x += 1
    return x
```

Outer loop goes through n times: **Θ(n)**

Inner loop goes through n times: **Θ(n)**

Everything else is constant. **Θ(1)**

- Computes n^2 very inefficiently
- Look at the loops. Are they **in terms of the input**?
    - Nested loops
    - Look at the ranges
    - Each iterating n times
- $\Theta(n) * \Theta(n) * \Theta(1) = \mathbf{\Theta(n^2)}$

QUADRATIC COMPLEXITY: EXAMPLE 2

- Decide if L1 is a subset of L2: are all elements of L1 in L2?

Yes:

```
L1 = [3, 5, 2]
L2 = [2, 3, 5, 9]
```

No:

```
L1 = [3, 5, 2]
L2 = [2, 5, 9]
```

```
def is_subset(L1, L2):
    for e1 in L1:
        matched = False
        for e2 in L2:
            if e1 == e2:
                matched = True
                break
        if not matched:
            return False
    return True
```

QUADRATIC COMPLEXITY: EXAMPLE 2

```
def is_subset(L1, L2):
    for e1 in L1:
        matched = False
        for e2 in L2:
            if e1 == e2:
                matched = True
                break
        if not matched:
            return False
    return True
```

Outer loop executed len(L1) times

Each iteration will execute inner loop up to len(L2) times

Θ(len(L1)*len(L2))

If L1 and L2 same length and none of elements of L1 in L2

Θ(len(L1)^2)

QUADRATIC COMPLEXITY: EXAMPLE 3

- Find intersection of two lists, return a list with each element appearing only once
  Example:

```
L1 = [3, 5, 2]
L2 = [2, 3, 5, 9]
returns [2,3,5]
```

```
L1 = [7, 7, 7]
L2 = [7, 7, 7]
returns [7]
```

```
def intersect(L1, L2):
    tmp = []
    for e1 in L1:
        for e2 in L2:
            if e1 == e2:
                tmp.append(e1)
    unique = []
    for e in tmp:
        if not(e in unique):
            unique.append(e)
    return unique
```

Build the list with common elements in L1 and L2. May have dups

Keep only unique values

QUADRATIC COMPLEXITY: EXAMPLE 3

```
def intersect(L1, L2):
    tmp = []
    for e1 in L1:
        for e2 in L2:
            if e1 == e2:
                tmp.append(e1)
    unique = []
    for e in tmp:
        if not(e in unique):
            unique.append(e)
    return unique
```

First nested loop takes **Θ(len(L1)*len(L2))** steps.

Second loop takes at most **Θ(len(L1)*len(L2))** steps. Typically not this bad.

- E.g: [7,7,7] and [7,7,7] makes tmp=[7,7,7,7,7,7,7,7,7]

Overall **Θ(len(L1)*len(L2))**

DIAMETER COMPLEXITY

Outer loop does len(L) passes:
Θ(len(L))

Inner loop does len(L) / 2 passes
(on average): **Θ(len(L))**

Everything else is constant **Θ(1)**

```
def diameter(L):
    farthest_dist = 0
    for i in range(len(L)):
        for j in range(i+1, len(L)):
            p1 = L[i]
            p2 = L[j]
            dist = math.sqrt( (p1[0]-p2[0])**2 + (p1[1]-p2[1])**2 )
            if dist > farthest_dist:
                farthest_dist = dist
    return farthest_dist
```

len(L) * len(L)/2 iterations = len(L)^2 / 2

Θ(len(L)^2)

YOU TRY IT!

```
def all_digits(nums):
    """ nums is a list of numbers """
    digits = [0,1,2,3,4,5,6,7,8,9]
    for i in nums:
        isin = False
        for j in digits:
            if i == j:
                isin = True
                break
        if not isin:
            return False
    return True
```

ANSWER:

What's the input?

Outer for loop is Θ(nums).

Inner for loop is Θ(1).

Overall: Θ(len(nums))

YOU TRY IT!

- Asymptotic complexity of f? And if L1,L2,L3 are same length?

```
def f(L1, L2, L3):
    for e1 in L1:
        for e2 in L2:
            if e1 in L3 and e2 in L3 :
                return True
    return False
```

ANSWER:

Θ(len(L1))* Θ(len(L2))* Θ(len(L3)+len(L3))

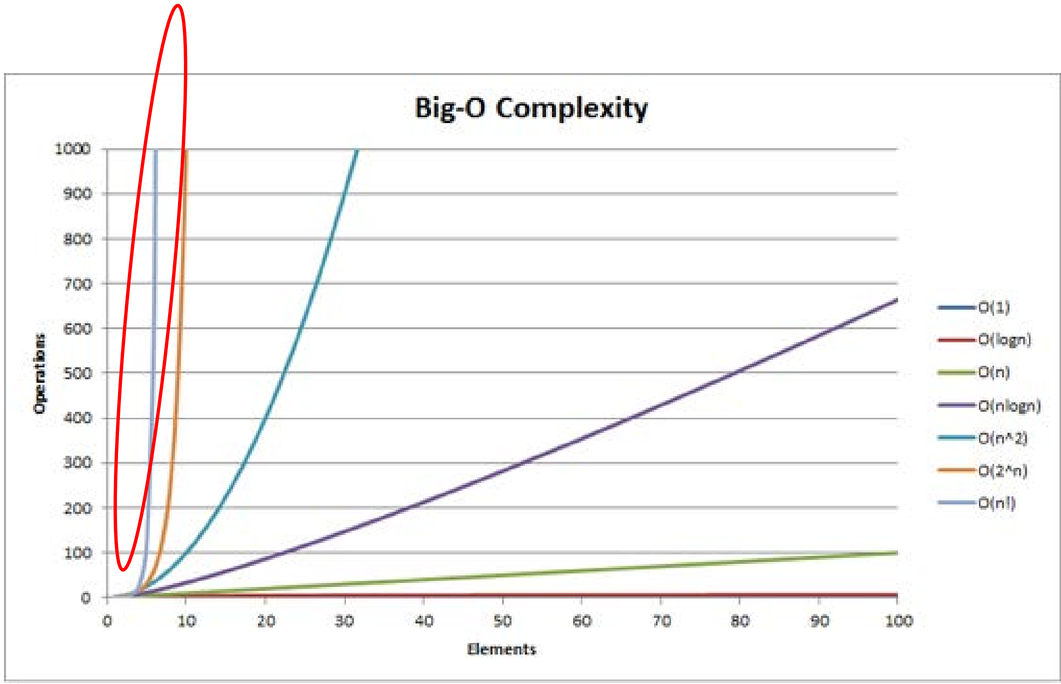
Overall: Θ(len(L1)*len(L2)*len(L3))

Overall if lists equal length: Θ(len(L1)**3)

EXPONENTIAL COMPLEXITY

EXPONENTIAL COMPLEXITY

- Recursive functions where have more than one recursive call for each size of problem
  - Fibonacci
- Many important problems are inherently exponential
  - Unfortunate, as cost can be high
  - Will lead us to consider approximate solutions more quickly



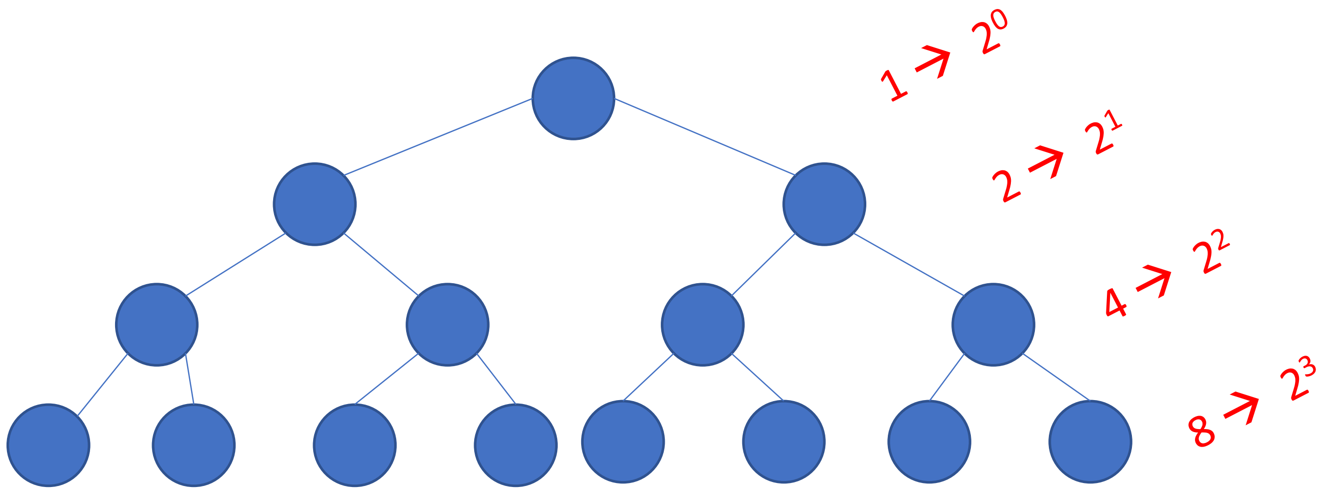
2^{30} ~= 1 million
2^{100} > # cycles than all the computers in the world working for all of recorded history could complete

COMPLEXITY OF RECURSIVE FIBONACCI

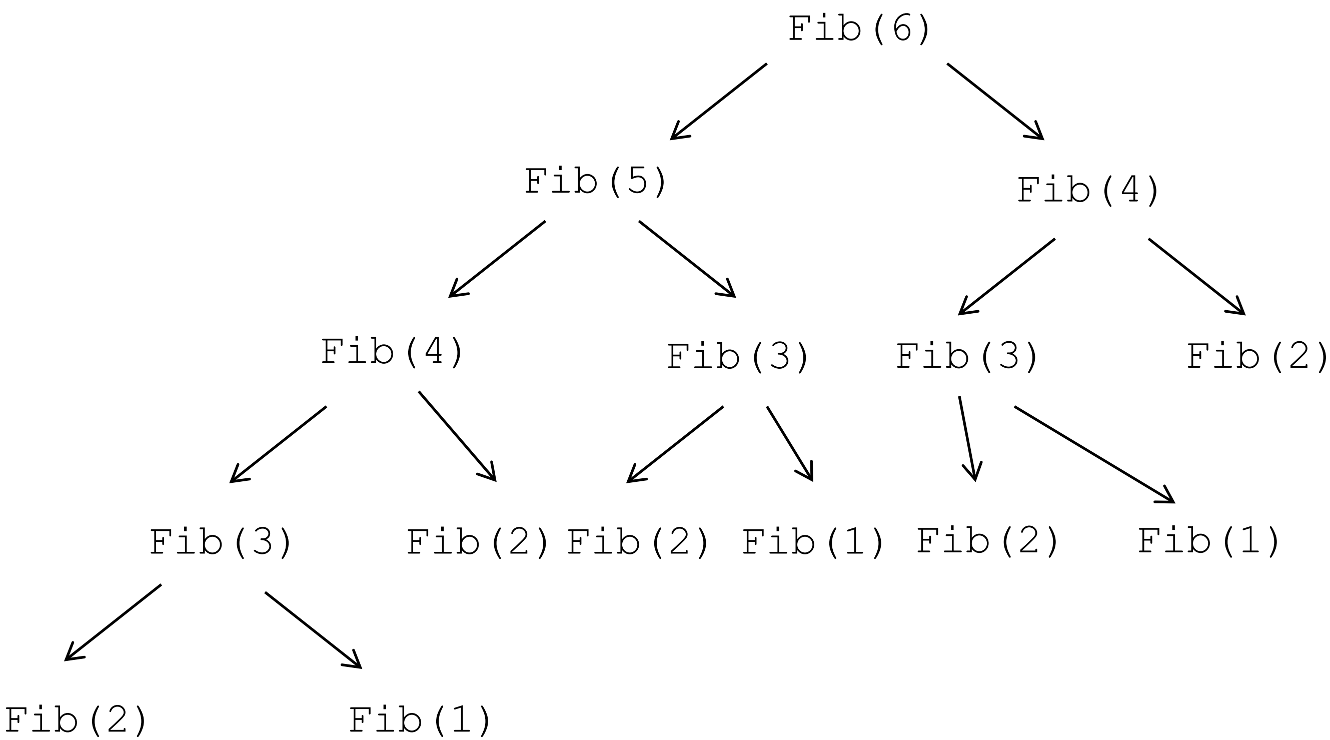
```
def fib_recur(n):
    """ assumes n an int >= 0 """
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return fib_recur(n-1) + fib_recur(n-2)
```

- **Worst case:**

  $\Theta(2^n)$



COMPLEXITY OF RECURSIVE FIBONACCI



- Can do a bit better than 2^n since tree thins out to the right
- But complexity is still order exponential

EXPONENTIAL COMPLEXITY: GENERATE SUBSETS

- Input is `[1, 2, 3]`
- Output is all combinations of elements of all lengths
  `[[],[1],[2],[3],[1,2],[1,3],[2,3],[1,2,3]]`

```
def gen_subsets(L):
    if len(L) == 0:
        return [[]]
    extra = L[-1:]
    smaller = gen_subsets(L[:-1])
    new = []
    for small in smaller:
        new.append(small+extra)
    return smaller+new
```

Base case: reach list of empty list

Create a list of just last element

All subsets without last element

For all smaller solutions, add one with last element

Combine those with last element and those without

VISUALIZING the ALGORITHM

Extra is [3]

[1,2,$\boxed{3}$]

Extra is [2]

[1,$\boxed{2}$]

Extra is [1]

[$\boxed{1}$]

Base case

[]

```
def gen_subsets(L):
    if len(L) == 0:
        return [[]]
    extra = L[-1:]
    smaller = gen_subsets(L[:-1])
    new = []
    for small in smaller:
        new.append(small+extra)
    return smaller+new
```

VISUALIZING the ALGORITHM

Extra is [3]

[1,2,3]

Extra is [2]

[1,2]

Extra is [1]

[1]

[[]]

Base case

[]

Returns

```
def gen_subsets(L):
    if len(L) == 0:
        return [[]]
    extra = L[-1:]
    smaller = gen_subsets(L[:-1])
    new = []
    for small in smaller:
        new.append(small+extra)
    return smaller+new
```

VISUALIZING the ALGORITHM

Extra is [3]

[1,2,3]

Extra is [2]

[1,2]

Extra is [1]

[1]

[]

[[],[1]]

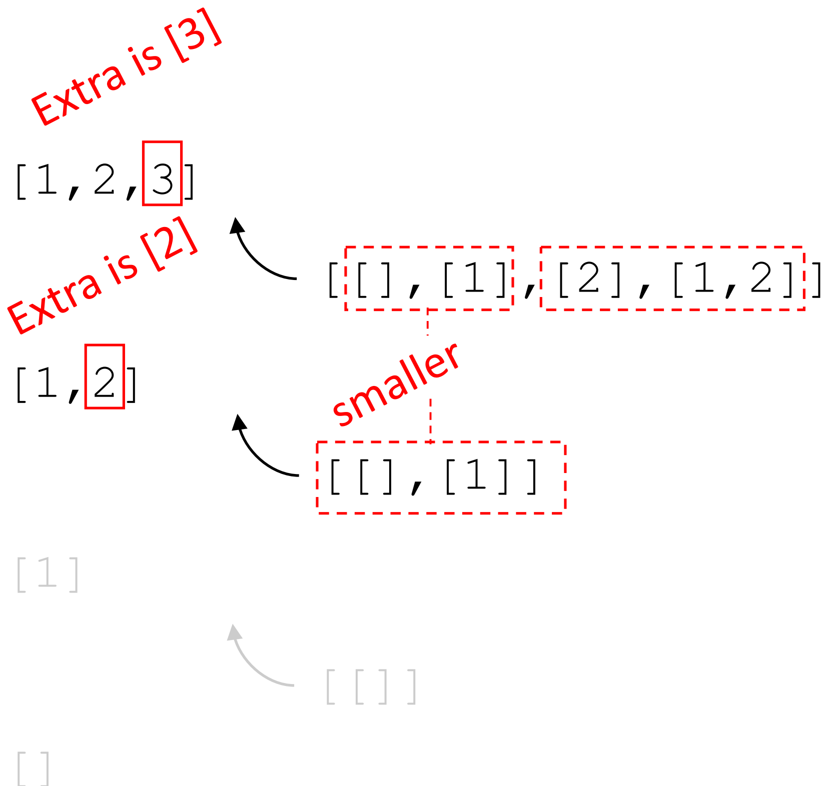
smaller

[[]]

Doubles smaller and returns

```python
def gen_subsets(L):
    if len(L) == 0:
        return [[]]
    extra = L[-1:]
    smaller = gen_subsets(L[:-1])
    new = []
    for small in smaller:
        new.append(small+extra)
    return smaller+new
```

VISUALIZING the ALGORITHM



Doubles smaller and returns

```
def gen_subsets(L):
    if len(L) == 0:
        return [[]]
    extra = L[-1:]
    smaller = gen_subsets(L[:-1])
    new = []
    for small in smaller:
        new.append(small+extra)
    return smaller+new
```

VISUALIZING the ALGORITHM

Extra is [3]

[[],[1],[2],[1,2],[3],[1,3],[2,3],[1,2,3]]

[1,2,3]

smaller

[[],[1],[2],[1,2]]

Doubles smaller and returns

[1,2]

[[],[1]]

[1]

[[]]

[]

```
def gen_subsets(L):
    if len(L) == 0:
        return [[]]
    extra = L[-1:]
    smaller = gen_subsets(L[:-1])
    new = []
    for small in smaller:
        new.append(small+extra)
    return smaller+new
```

VISUALIZING the ALGORITHM

[[],[1],[2],[1,2],[3],[1,3],[2,3],[1,2,3]]

[1,2,3]

[1,2]

[[],[1]]

[1]

[[]]

[]

```
def gen_subsets(L):
    if len(L) == 0:
        return [[]]
    extra = L[-1:]
    smaller = gen_subsets(L[:-1])
    new = []
    for small in smaller:
        new.append(small+extra)
    return smaller+new
```

EXPONENTIAL COMPLEXITY GENERATE SUBSETS

```
def gen_subsets(L):
    if len(L) == 0:
        return [[]]
    extra = L[-1:]
    smaller = gen_subsets(L[:-1])
    new = []
    for small in smaller:
        new.append(small+extra)
    return smaller+new
```

- Assuming append is constant time
- Time to make sublists includes time to solve **smaller problem**, and time needed to **make a copy** of all elements in smaller problem

EXPONENTIAL COMPLEXITY GENERATE SUBSETS

```
def gen_subsets(L):
    if len(L) == 0:
        return [[]]
    extra = L[-1:]
    smaller = gen_subsets(L[:-1])
    new = []
    for small in smaller:
        new.append(small+extra)
    return smaller+new
```

- Think about **size of smaller**
  - For a set of size k there are 2^k cases, doubling the size every call
  - So to solve need $2^{n-1} + 2^{n-2} + \ldots + 2^0$ steps = $\Theta(2^n)$
- Time to **make a copy of smaller**
  - Concatenation isn't constant
  - $\Theta(n)$
- Overall complexity is **$\Theta(n*2^n)$ where n=len(L)**

LOGARITHMIC COMPLEXITY

TRICKY COMPLEXITY

```
def digit_add(n):
    """ assume n an int >= 0 """
    answer = 0
    s = str(n)
    for c in s[::-1]:
        answer += int(c)
    return answer
```

Linear Θ(len(s))
Loops through the length of n as a str

But what in terms of input n?

- Adds digits of a number together
  - n = 83, but the loop only iterates 2 times. Relationship?
  - n = 4271, but the loop only iterates 4 times! Relationship??

4	2	7	1

First time through loop, extract the least significant digit

1

TRICKY COMPLEXITY

```
def digit_add(n):
    """ assume n an int >= 0 """
    answer = 0
    s = str(n)
    for c in s[::-1]:
        answer += int(c)
    return answer
```

Linear Θ(len(s))

But what in terms of input n?

- Adds digits of a number together
  - n = 83, but the loop only iterates 2 times. Relationship?
  - n = 4271, but the loop only iterates 4 times! Relationship??

Second time through loop, extract the next least significant digit

4	2	7

7 + 1

TRICKY COMPLEXITY

```python
def digit_add(n):
    """ assume n an int >= 0 """
    answer = 0
    s = str(n)
    for c in s[::-1]:
        answer += int(c)
    return answer
```

Linear Θ(len(s))

But what in terms of input n?

- Adds digits of a number together
  - n = 83, but the loop only iterates 2 times. Relationship?
  - n = 4271, but the loop only iterates 4 times! Relationship??

4 2

Third time through loop, extract the next least significant digit

2 + 7 + 1

TRICKY COMPLEXITY

```python
def digit_add(n):
    """ assume n an int >= 0 """
    answer = 0
    s = str(n)
    for c in s[::-1]:
        answer += int(c)
    return answer
```

Linear Θ(len(s))

But what in terms of input n?

- Adds digits of a number together
  - n = 83, but the loop only iterates 2 times. Relationship?
  - n = 4271, but the loop only iterates 4 times! Relationship??

4

Last time through loop, extract the next least significant digit

4 + 2 + 7 + 1

TRICKY COMPLEXITY

```
def digit_add(n):
    """ assume n an int >= 0 """
    answer = 0
    s = str(n)
    for c in s[::-1]:
        answer += int(c)
    return answer
```

Linear **Θ(len(s))**

But what in terms of input n?

- Adds digits of a number together
- Tricky part: iterate over **length of string**, not magnitude of n
  - Think of it like dividing n by 10 each iteration
  - $n/10^{len(s)} = 1$ (i.e. divide by 10 until there is 1 element left to add)
  - len(s) = log(n)
- **Θ(log n)** – base doesn't matter

LOGARITHMIC COMPLEXITY

- Complexity grows as log of size of one of its inputs
- Example algorithm: **binary search** of a list
- Example we'll see in a few slides: one **bisection search** implementation

LIST AND DICTIONARIES

- Must be **careful** when using built-in functions!

Lists – n is len(L)

- index $\Theta(1)$
- store $\Theta(1)$
- length $\Theta(1)$
- append $\Theta(1)$
- == $\Theta(n)$
- remove $\Theta(n)$
- copy $\Theta(n)$
- reverse $\Theta(n)$
- iteration $\Theta(n)$
- in list $\Theta(n)$

Dictionaries – n is len(d)

- index $\Theta(1)$
- store $\Theta(1)$
- length $\Theta(1)$
- delete $\Theta(1)$
- .keys $\Theta(n)$
- .values $\Theta(n)$
- iteration $\Theta(n)$

SEARCHING ALGORITHMS

SEARCHING ALGORITHMS

- Linear search
  - **Brute force** search
  - List does not have to be sorted
- Bisection search
  - List **MUST be sorted** to give correct answer
  - Will see two different implementations of the algorithm

LINEAR SEARCH
ON UNSORTED LIST

```
def linear_search(L, e):
    found = False
    for i in range(len(L)):
        if e == L[i]:
            found = True
    return found
```

The loop goes through len(L): **Θ(len(L))**

Everything else is constant. **Θ(1)**

- Must look through all elements to decide it's not there
- **Θ(len(L))** for the loop * **Θ(1)** to test if e == L[i]
- Overall complexity is **Θ(n) where n is len(L)**
- **Θ(len(L))**

LINEAR SEARCH ON UNSORTED LIST

```
def linear_search(L, e):

    for i in range(len(L)):
        if e == L[i]:
            return True
    return False
```

Speed up a little by returning True here, but speed up doesn't impact worst case

- Must look through all elements to decide it's not there
- **Θ(len(L))** for the loop * **Θ(1)** to test if e == L[i]
- Overall complexity is **Θ(n) where n is len(L)**
- **Θ(len(L))**

LINEAR SEARCH ON **SORTED** LIST

```
def search(L, e):
    for i in L:
        if i == e:
            return True
        if i > e:
            return False
    return False
```

The loop goes through len(L):
Θ(len(L))

Everything else is constant.
Θ(1)

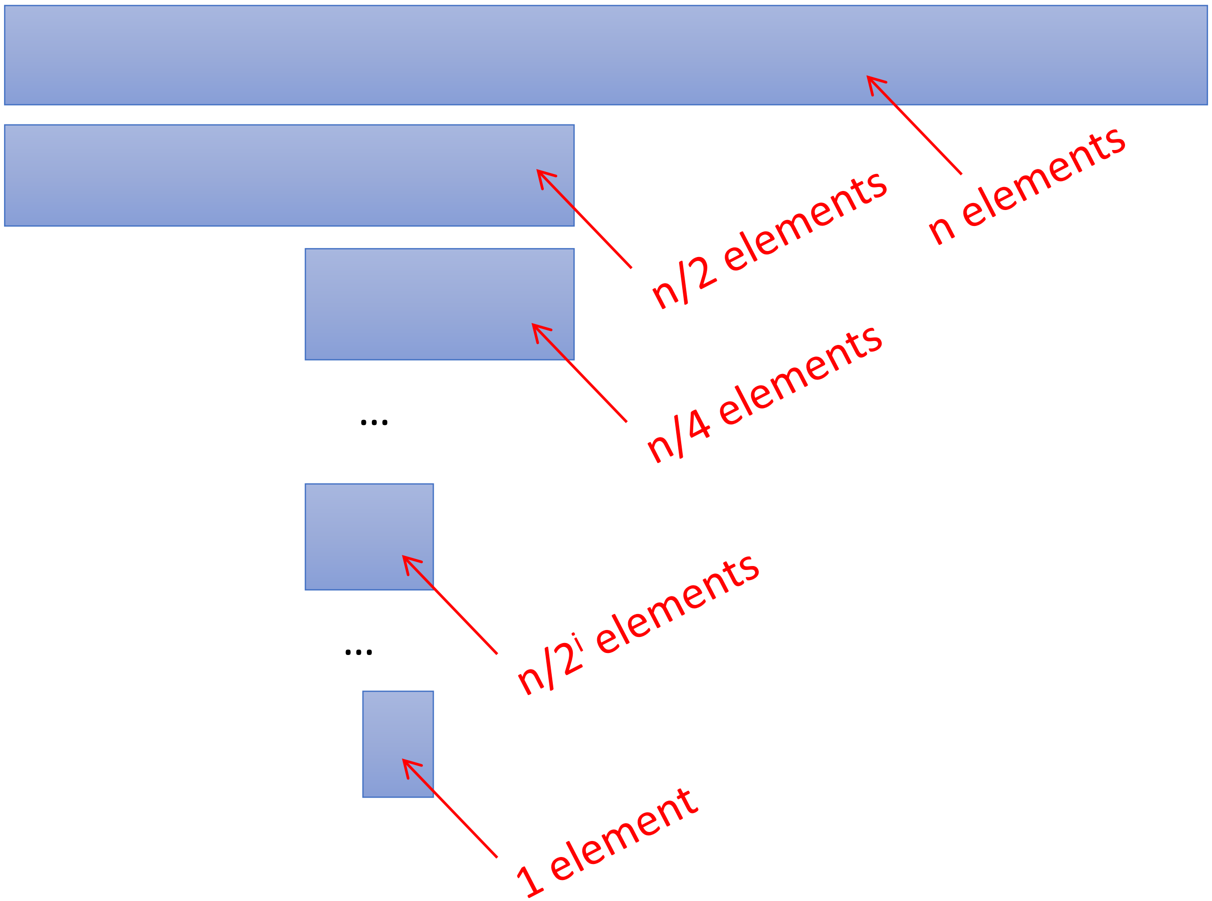
- Must only look until reach a number greater than e
- **Θ(len(L))** for the loop * **Θ(1)** to test if i == e or i > e
- Overall complexity is **Θ(len(L))**
  Θ(n) where n is len(L)

BISECTION SEARCH FOR AN ELEMENT IN A SORTED LIST

1) Pick an index, `i`, that divides list in half
2) Ask if `L[i] == e`
3) If not, ask if `L[i]` is larger or smaller than `e`
4) Depending on answer, search left or right half of `L` for `e`

- A new version of **divide-and-conquer: recursion!**
- Break into smaller versions of problem (smaller list), plus simple operations
- Answer to smaller version is answer to original version

BISECTION SEARCH COMPLEXITY ANALYSIS



- Finish looking through list when

  $1 = n/2^i$

- So… relationship between original length of list and how many times we divide the list: $i = \log n$

- Complexity is **Θ(log n) where n is len(L)**

BIG IDEA

Two different implementations have two different Θ values.

BISECTION SEARCH IMPLEMENTATION 1

```
def bisect_search1(L, e):
    if L == []:
        return False
    elif len(L) == 1:
        return L[0] == e
    else:
        half = len(L)//2
        if L[half] > e:
            return bisect_search1( L[:half], e)
        else:
            return bisect_search1( L[half:], e)
```

constant
Θ(1)

constant
Θ(1)

constant
Θ(1)

NOT constant,
copies list with
each function call

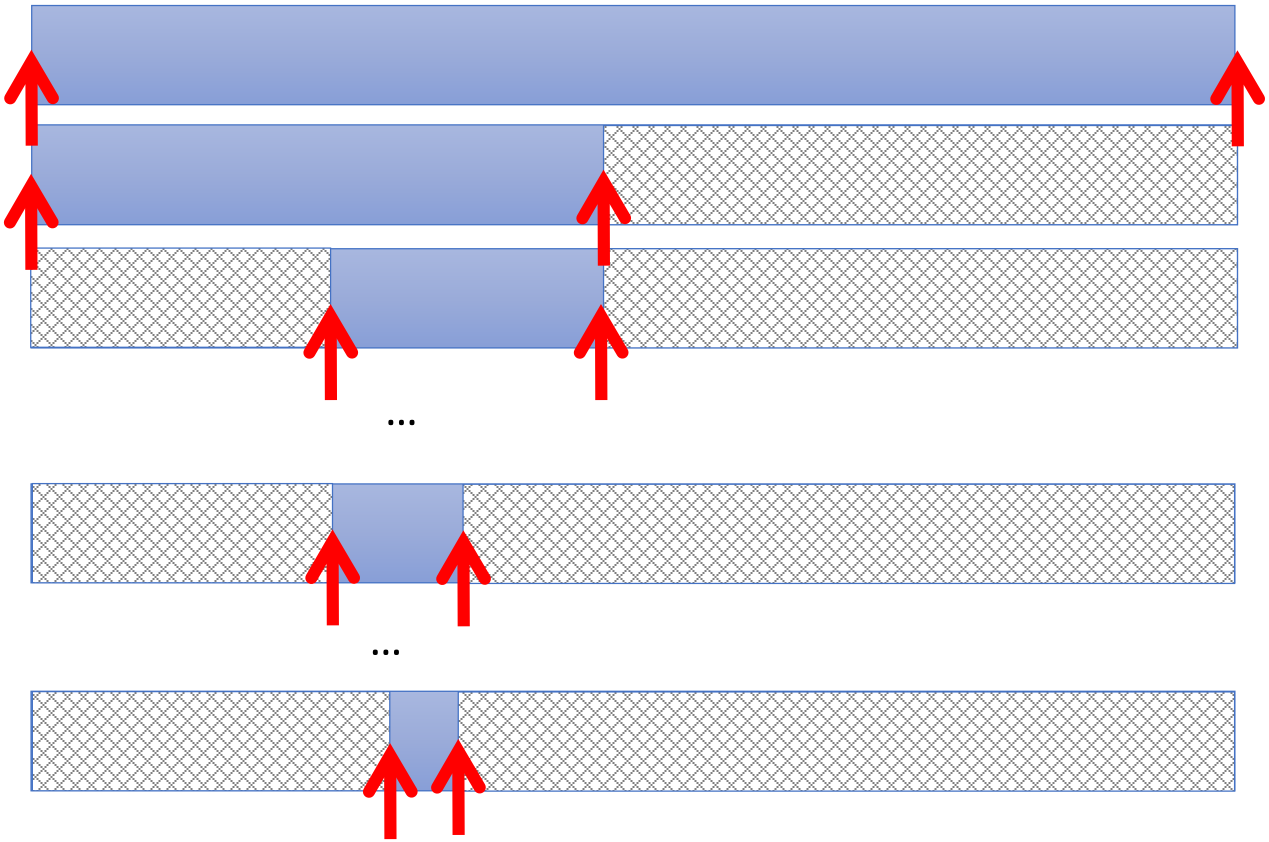
NOT constant
Θ(log(len(L))

NOT constant
Θ(log(len(L))

COMPLEXITY OF bisect_search1 (where n is len(L))

- **Θ(log n)** bisection search calls
  - Each recursive call cuts range to search in half
  - Worst case to reach range of size 1 from n is when $n/2^k = 1$ or when $k = \log n$
  - We do this to get an expression relating k to n
- **Θ(n)** for each bisection search call to copy list
  - Cost to set up recursive call at each level of recursion
- Θ(log n) * Θ(n) = **Θ(n log n) where n = len(L)**
  ^ this is the answer in this class
- If careful, notice list is also halved on each recursive call
  - Infinite series (don't worry about this in this class)
  - Θ(n) is a tighter bound because copying list dominates log n

BISECTION SEARCH ALTERNATE IMPLEMENTATION



- Reduce size of problem by factor of 2 each step
- Keep track of low and high indices to search list
- Avoid copying list
- Complexity of recursion is **Θ(log n) where n is len(L)**

BISECTION SEARCH IMPLEMENTATION 2

Instead of copying the list, keep track of the low and high list indices

```
def bisect_search2(L, e):
    def bisect_search_helper(L, e, low, high):
        if high == low:
            return L[low] == e
        mid = (low + high)//2
        if L[mid] == e:
            return True
        elif L[mid] > e:
            if low == mid: #nothing left to search
                return False
            else:
                return bisect_search_helper(L, e, low, mid - 1)
        else:
            return bisect_search_helper(L, e, mid + 1, high)
    if len(L) == 0:
        return False
    else:
        return bisect_search_helper(L, e, 0, len(L) - 1)
```

NOT constant
Θ(log(len(L))

NOT constant
Θ(log(len(L))

Kick off the recursive helper

COMPLEXITY OF bisect_search2 and helper (where n is len(L))

- **Θ(log n)** bisection search calls
  - Each recursive call cuts range to search in half
  - Worst case to reach range of size 1 from n is when $n/2^k = 1$ or when $k = \log n$
  - We do this to get an expression relating k to n
- Pass list and indices as parameters
  - List never copied, just re-passed
  - **Θ(1)** on each recursive call
- Θ (log n) * Θ(1) = **Θ(log n) where n is len(L)**

WHEN TO SORT FIRST AND THEN SEARCH?

SEARCHING A SORTED LIST
-- n is len(L)

- Using **linear search**, search for an element is **Θ(n)**
- Using **binary search**, can search for an element in **Θ(log n)**
  - Assumes the **list is sorted**!
- When does it make sense to **sort first then search**?

Time to sort

Time for binary search

Time for linear search

- $\boxed{\text{SORT}} + \boxed{\Theta(\texttt{log n})} < \boxed{\Theta(\texttt{n})}$
  implies that SORT < $\Theta(\texttt{n}) - \Theta(\texttt{log n})$

- When is sorting is less than $\Theta(\texttt{n})$??!!?
  → Never true because you'd at least have to look at each element!

AMORTIZED COST
-- n is len(L)

- Why bother sorting first?
- **Sort a list once** then do **many searches**
- **AMORTIZE cost** of the sort over many searches

Only once! Do K searches

- $\boxed{\text{SORT}} + \boxed{\texttt{K}} * \Theta(\texttt{log n}) < \boxed{\texttt{K}} * \Theta(\texttt{n})$

  implies that for large `K`, **SORT time becomes irrelevant**

COMPLEXITY CLASSES SUMMARY

- Compare efficiency of algorithms
- Lower order of growth
- Using **Θ for an upper and lower ("tight") bound**

- Given a function f:
  - Only look at **items in terms of the input**
  - Look at **loops**
    - Are they in terms of the input to f?
    - Are there nested loops?
  - Look at **recursive calls**
    - How deep does the function call stack go?
  - Look at **built-in functions**
    - Any of them depend on the input?

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